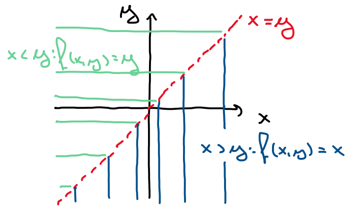
cv11

8.4) Kde je funkce $f(x,y) = \max\{x,y\}$ spojitě diferencovatelná?



$$\frac{\partial f}{\partial x} = \begin{cases} 1, & x > y \\ 0, & x < y \\ \text{neexistuje}, & x = y \end{cases} \qquad \frac{\partial f}{\partial y} = \begin{cases} 0, & x > y \\ 1, & x < y \\ \text{neexistuje}, & x = y \end{cases}$$

$f(x,y)$ je spojitě diferencovatelná na $\mathbb{R}^2 \setminus \{(x,y) \mid x = y\}$

8.5) Najděte totální derivaci (Jacobiho matici)

1) $f(x) = x^2$ $\qquad f'(x) = [2x]$

2) $f(x,y) = x^2 - y^2$ $\qquad f'(x,y) = [2x \;\; -2y]$

3) $f(\vec{x}) = a$ $\qquad f'(\vec{x}) = [0\,0 \dots 0]$

4) $f(\vec{x}) = x_1$ $\qquad f'(\vec{x}) = [1\,0 \dots 0]$

5) $f(\vec{x}) = \vec{a}^T\vec{x}$ $\qquad f'(\vec{x}) = \vec{a}^T$

6) $f(\vec{x}) = \vec{a}^T\vec{x} + b$ $\qquad f'(\vec{x}) = \vec{a}^T$

7) $f(\vec{x}) = \vec{x}^T A \vec{x} + \vec{b}^T\vec{x} + c$ $\qquad f'(\vec{x}) = \vec{x}^T(A + A^T) + \vec{b}^T$

$f(\vec{x}) = \vec{x}^T A \vec{x} = \sum_i \sum_j a_{ij} x_i x_j$;

$$\frac{\partial f(\vec{x})}{\partial x_k} = \frac{\partial}{\partial x_k}\left[\sum_i \sum_j a_{ij} x_i x_j\right] = \frac{\partial}{\partial x_k}\left[\underbrace{\sum_{i \neq k} \sum_{j \neq k} a_{ij} x_i x_j}_{i \neq k,\, j \neq k} + \underbrace{\sum_{j \neq k} a_{kj} x_k x_j}_{i = k,\, j \neq k} + \underbrace{\sum_{i \neq k} a_{ik} x_i x_k}_{i \neq k,\, j = k} + \underbrace{a_{kk} x_k x_k}_{i = k,\, j = k}\right]$$

$$= 0 + \sum_{j \neq k} a_{kj} x_j + \sum_{i \neq k} a_{ik} x_i + 2a_{kk}x_k = \sum_j a_{kj} x_j + \sum_i a_{ik} x_i = (A^T x)_k + (Ax)_k$$

může nabývat i k

$$f'(\vec{x}) = \left[\frac{\partial f}{\partial x_1} \;\cdots\; \frac{\partial f}{\partial x_n}\right] = x^T(A + A^T)$$

8) $f(\vec{x}) = e^{-\vec{x}^T\vec{x}}$ $\qquad f'(\vec{x}) = -2e^{-\vec{x}^T\vec{x}} \cdot \vec{x}^T$

10) $f(t) = (\cos t, \sin t)$ $\qquad f'(t) = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$

11) $f(t) = (\cos t, \sin t, at)$ $\qquad f'(t) = \begin{bmatrix} -\sin t \\ \cos t \\ a \end{bmatrix}$

12) $f(t) = \vec{x} + t\vec{v}$ $\qquad f'(t) = \vec{v}$

13) $f(\vec{x}) = \vec{x}$ $\qquad f'(\vec{x}) = I$

14) $f(\vec{x}) = \vec{a}$ $\qquad f'(\vec{x}) = 0$

15) $f(\vec{x}) = A\vec{x}$ $\qquad f'(\vec{x}) = A$

8.6) $f(x,y)$ je diferencovatelná funkce dvou proměnných

a) spočítej derivaci f podle polárních souřadnic (r, φ) kde $x = r \cdot \cos\varphi$ $y = r \cdot \sin\varphi$

pomocí pravidla pro složené derivace $\frac{dg(f(x))}{dx} = g'(f(x))\, f'(x)$

$f: \mathbb{R}^2 \to \mathbb{R}$; $g: \mathbb{R}^2 \to \mathbb{R}^2$, $g\left(\begin{bmatrix} r \\ \varphi \end{bmatrix}\right) = \begin{bmatrix} r\cos\varphi \\ r\sin\varphi \end{bmatrix}$

$$\underbrace{\frac{d f(g(r,\varphi))}{d(r,\varphi)}}_{\mathbb{R}^{1\times 2}} = \underbrace{f'(g(r,\varphi))}_{\mathbb{R}^{1\times 2}} \cdot \underbrace{\frac{d g(r,\varphi)}{d(r,\varphi)}}_{\mathbb{R}^{2\times 2}} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial r} & \frac{\partial g_1}{\partial \varphi} \\ \frac{\partial g_2}{\partial r} & \frac{\partial g_2}{\partial \varphi} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x}\cos\varphi + \frac{\partial f}{\partial y}\sin\varphi & -\frac{\partial f}{\partial x} r\sin\varphi + \frac{\partial f}{\partial y} r\cos\varphi \end{bmatrix}$$

8.8) Odvoďte co nejjednodušší vzorec pro Jacobiho matici

b) $f(x) = \|x\| = \sqrt{x^T x}$

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\sqrt{\Sigma_j x_j^2}\right] = \frac{\partial}{\partial x_i}\left[\left(\Sigma_j x_j^2\right)^{\frac{1}{2}}\right] = \underbrace{\frac{1}{2}\left(\Sigma_j x_j^2\right)^{-\frac{1}{2}}}_{(\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}}} \cdot \underbrace{2x_i}_{\frac{\partial}{\partial x_i}\Sigma_j x_j^2 = \frac{\partial}{\partial x_i} x_i^2 = 2x_i} = \frac{x_i}{\sqrt{\Sigma_j x_j^2}} = \frac{x_i}{\|x\|}$$

$$f'(x) = \left[\frac{\partial f}{\partial x_1} \ \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n}\right] = \frac{x^T}{\|x\|}$$

d) $f(x) = \|g(x)\|$ $\quad f'(x) = \underbrace{\frac{g(x)^T}{\|g(x)\|}}_{z\ b)} g'(x)$

nebo

$$\frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i}\left[\sqrt{g(x)^T g(x)}\right] = \frac{1}{2}\left[g(x)^T g(x)\right]^{-\frac{1}{2}} \cdot 2g(x)^T \cdot \frac{\partial g(x)}{\partial x_i}$$

8.14) Čemu je roven Taylorův polynom 1. st. polynomu 1. st. ?
A 2. st. polynomu 2. st. ?

a) $f(x) = ax + b$

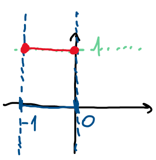
$T^1_{x_0}(x) = f(x_0) + f'(x_0)(x - x_0)$
$= ax_0 + b + \quad a\cdot(x - x_0) = ax + b \quad \Rightarrow$ jsou si rovny

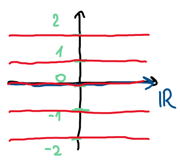
b) analogické a), jsou si rovny

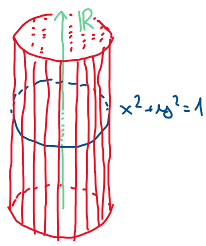
pozn.: Taylorův polynom stupně k v bodě x má všechny derivace až do řádu k stejné jako funkce f
$\Rightarrow$ pokud aproximujeme polynom až k-tého stupně, budou všechny derivace stejné a Taylorův polynom bude identický originálnímu polynomu

9.2) Načrtněte následující množiny

a) $\underline{[-1, 0]} \times \underline{\{1\}}$ $\qquad$ c) $\underline{\mathbb{R}} \times \underline{\mathbb{Z}}$ $\qquad$ e) $\underline{\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}} \times \underline{\mathbb{R}}$

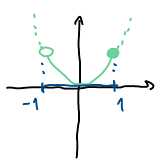






9.3) Co je vnitřek a hranice množin

f) $M = \{(x,y) \in \mathbb{R}^2 \mid y = x^2, -1 < x \le 1\}$



$\text{int}\, M = \emptyset$
$\partial M = M \cup \{(-1,1)\}$